Journal of Engineering Mathematics **46:** 281–297, 2003.

The biharmonic problem and progress in the development of analytical methods for the solution of boundary-value problems

V. T. GRINCHENKO
Institute of Hydromechanics, National Academy of Sciences, 03057 Kiev, Ukraine. e-mail: vin-igm@gu.kiev.ua

Received 4 April 2003; accepted in revised form 5 May 2003

Abstract. A comparative analysis of harmonic and biharmonic boundary-value problems for 2D problems on a rectangle is given. Some common features of two types of problems are emphasized and special attention is given to the basic distinction between them. This distinction was thoroughly studied for the first time by L. N. G. Filon with respect to some plane problems in the theory of elasticity. The analysis permits to introduce an important aspect of the general solution of boundary-value problems. The procedure for solving the biharmonic problem involves both the method of homogenous solutions and the method of superposition. For some cases involving self-equilibrated loadings on one pair of sides of the rectangle, the complete solution, including calculation of the quantitative characteristics of the displacements and stresses, is given. The efficiency of the numerical implementation of the general solutions is shown. The analysis of the quantitative data allows to elucidate some main points of the Saint-Venant principle.

Key words: elastic rectangle, general solution of biharmonic boundary-value problem, method of homogenous solutions, method of superposition, Saint-Venant principle.

1. Introduction

Despite the increasing role of numerical methods for solving boundary-value problems arising in the mathematical modelling of engineering problems, the development of analytical approaches to the solution of complex boundary-value problems remains important in engineering mathematics. In the long history of mathematical physics the study of boundary-value problems for the Laplace equation (static harmonic problems) and the Helmholtz equation (wave harmonic problems) has played a special role. In the development of methods for the solution of such boundary-value problems, general approaches to the construction and analysis of analytical solutions of boundary-value problems were given. An orderly theory of higher transcendental functions was created for the analytical representation of solutions of other types of boundary-value problems [1, Chapter 10].

However, the transition to problems described by partial differential equations of higher order frequently leads to particular mathematical difficulties. The clearest illustration of this statement is perhaps given by a comparative analysis of harmonic and biharmonic boundary-value problems. Some results of such an analysis will be presented in this paper, basically, with reference to two-dimensional problems. The famous Filon paper [2] was the first to attract the attention of the scientific community to the important difference between the two classes of boundary-value problems. The never-decreasing interest in boundary-value problems for the biharmonic equation can be attributed to a series of reasons. Within the framework of problems in classical mathematical physics, a comparative analysis of harmonic and biharmonic problems illustrates those principal mathematical difficulties which result from increasing the order of the equation.

At first it seems that boundary-value problems for the biharmonic equation can be solved by the same methods as developed for harmonic problems. This belief motivated Kolosov [3] and Muskhelishvili [4] to present their highly acclaimed solution of the biharmonic equation in terms of two analytic (harmonic) functions. The central idea of the very popular book by Muskhelishvili consists in underlining the likeness of harmonic and biharmonic problems. The classical theory of analytical functions leads to a practically complete solution of a general harmonic boundary-value problem.

Although in many respects the theory of biharmonic problems in terms of the theory of analytical functions is quite elegant, a consideration of boundary-value problems for rather simple domains has shown a sharp distinction between these two types of problems. The meaning of this general statement becomes especially clear when comparing difficulties of the solution of 2D harmonic and biharmonic boundary-value problems on a rectangular domain. These problems will be used to illustrate one of the basic ideas developed for the construction of the solution of biharmonic boundary-value problems. A complete review of the history of the different approaches to biharmonic boundary-value-problem solutions is given by Meleshko [5].

Historically, the interest in biharmonic problems was also stimulated by important engineering problems. Brilliant calculations of the stresses in a clamped rectangular elastic plate were done by the Russian naval architect Bubnov [6]. The analysis of the results of these calculations has motivated the development of mathematical methods for biharmonic problems. Stimulated by the solution of boundary-value problems for the biharmonic equation [7], a development of the theory of infinite systems of linear algebraic equations has turned out to be extremely important for the development of methods of solution for many problems in mathematical physics. Indeed, in solving biharmonic boundary-value problems, new concepts in the theory of boundary-value problems for partial differential equations have been formed. One such new concept, namely the general solution of boundary-value problems will be considered in this paper.

The paper is organized as follows. The analysis of the procedure used to obtain the complete solution of the harmonic problem on a rectangle is given in Section 2. It is shown that the solution of this problem can be presented in various forms, which can be used to satisfy the boundary conditions. In Section 3 properties of different systems of partial solutions of the biharmonic equation are considered. The solutions, giving potentially the possibility to get the complete solution for the boundary-value problem on a rectangle, are constructed. The corresponding choice of such solutions gives a method for constructing the general solution of the boundary-value problem. To analyse the properties of such solutions, a short description of homogenous solutions is given in Section 4. A numerical implementation of the general solution for concrete boundary conditions (Section 5) permits to discuss important features of the method. Specific numerical data are of interest to the understanding of important peculiarities of the Saint-Venant principle.

2. Analysis of the solution of the harmonic boundary-value problem on a rectangle

The main point of the problems tackled by Filon, by considering boundary-value problems for an elastic half-strip or a finite cylinder, should be seen in the light of the history of the

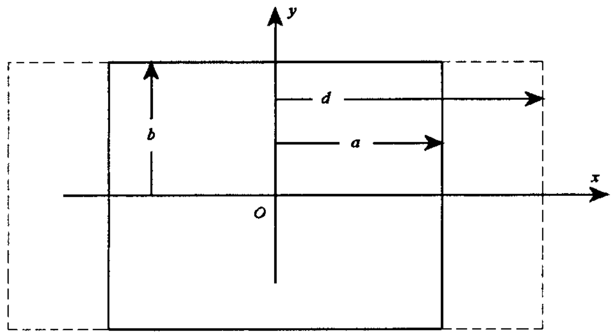

Figure 1. The domain geometry and choice of the coordinate system. Boundaries of the domain are shown by solid lines. Dashed lines along with part of the boundaries of the given domain form another rectangle which encloses the given area.

development of methods of mathematical physics. Prior to Filon, attention was given mainly to the development of methods for solving boundary-value problems in potential theory. The Sturm-Liouville theory for systems of eigenfunctions of second-order differential operators had been developed for that purpose. Use of such functions provides a methodology to obtain elegant closed solutions of boundary-value problems in potential theory in the separable coordinate systems for canonical domains. For these domains their boundary is formed by parts of coordinate surfaces (lines). To construct closed-form solutions of harmonic boundary-value problems for canonical domains, a special procedure is used. The specific features of this procedure are illustrated by a 2D example.

To show some important features for a subsequent generalization of the solution of harmonic problems, let us consider a very simple two-dimensional Dirichlet problem for the Laplace equation in a rectangular domain. The geometry of the domain and the coordinate system are shown in Figure 1.

It is necessary to find a function $\varphi(x, y)$ satisfying the Laplace equation

$$\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 0 \tag{1}$$

in the domain $-a \le x \le a$, $-b \le y \le b$ and boundary conditions at the boundaries of the rectangle

$$\varphi(x, \pm b) = F_1(x), \qquad \varphi(\pm a, y) = F_2(y). \tag{2}$$

To illustrate the basic features of the classical approach towards the solution of the boundary-value problem, it is sufficient to consider the case of a function φ, that is even in both coordinates, with even functions in the boundary conditions (2): $F_1(x) = F_1(-x)$, $F_2(y) = F_2(-y)$.

A remarkable property of the harmonic equation (1) is the possibility to represent its solution in the form of a product of functions of x and y. This is a remarkable property which many equations of mathematical physics have in so-called separable coordinate systems. According to the procedure of separation of variables, we shall present the harmonic function sought as a product $\varphi(x, y) = X(x)Y(y)$. For some arbitrary constant λ^2 it is possible to obtain from Equation (1) two separate equations for the required functions

$$\frac{\mathrm{d}^2X}{\mathrm{d}x^2} - \lambda^2 X = 0, \qquad \frac{\mathrm{d}^2Y}{\mathrm{d}y^2} + \lambda^2 Y = 0. \tag{3}$$

The solutions of these simple equations are well known. It is important to realize that the product of the functions $X(x)$ and $Y(y)$ provides a particular solution of the harmonic Equation (1) for arbitrary (real, imaginary or complex) values of the constant λ^2. This approach allows one to construct sets of particular solutions by a choice of special values of the separation constant. Now one can propose a constructive way to obtain the solution of the boundary-value problem (2). We have to construct such sets of particular solutions of the harmonic equation, which undoubtedly allow us to satisfy the boundary conditions (2). As the boundary conditions are defined for constant values of one of the coordinates, this problem is simply solved for many domains within the framework of the Sturm-Liouville theory.

Let us consider the boundary conditions at $y = \pm b$. To satisfy the corresponding conditions in (2) the harmonic function sought must contain at $y = \pm b$ the complete set of functions with arbitrary coefficients. It is not difficult to construct such a solution. Taking for λ in Equations (3) the following values

$$\lambda = \mathrm{i}\alpha_n, \qquad \alpha_n = \frac{n\pi}{l}, \qquad l > a, \tag{4}$$

we obtain the solution of the harmonic equation as

$$\varphi_I = \sum_{n=0}^{\infty} A_n \cos \alpha_n x \cosh \alpha_n y. \tag{5}$$

Here A_n are arbitrary constants. The symmetry properties of the field were taken into account when constructing particular solutions of Equations (3).

It is obvious that expression (5) satisfies term by term the harmonic equation and includes a complete set of functions at the surfaces $y = \pm b$, $a \le x \le a$. Choosing proper values of the arbitrary constants A_n, we can satisfy the boundary conditions on the sides $y = \pm b$ of rectangle. The procedure for the actual determination of the constants A_n will be especially simple for the following two values of the separation constant $\alpha_n = n\pi/a$ or $\alpha_n = (2n + 1)\pi/(2a)$. For these two cases the sets of functions $\cos \alpha_n x$ are not only complete, but also orthogonal on the interval $-a \le x \le a$. The two series (5) corresponding to two given sequences of separation constants α_n have important properties. They not only give the possibility to satisfy boundary conditions on the surfaces $y = \pm b$, but also satisfy homogeneous boundary conditions at the surfaces $x = \pm a$, correspondingly for the function or its normal derivative. Such special solutions of the boundary-value problems were called homogeneous solutions.

One can see that the series

$$\varphi_{II} = \sum_{m=0}^{\infty} B_m \cos \beta_m y \cosh \beta_m x \tag{6}$$

is a solution of the harmonic equation which can satisfy boundary conditions on the sides $x = \pm a$. As to β_m one can repeat the comments concerning the parameters α_n. The best way to simplify the solution of the boundary-value problem is to use the homogeneous solution with $\beta_m = m\pi/b$. Thus, the general solution of the boundary-value problem (2) is the sum of the two homogeneous solutions, namely

$$\varphi = \varphi_I + \varphi_{II} = \sum_{n=0}^{\infty} A_n \cos \alpha_n x \cosh \alpha_n y + \sum_{m=0}^{\infty} B_m \cos \beta_m y \cosh \beta_m x,$$

$$(\alpha_n = n\pi/a, \quad \beta_m = m\pi/b). \tag{7}$$

In this case actual fulfillment of the boundary conditions is reduced to the solution of a sequence of algebraic equations with one unknown:

$$\begin{aligned} A_0 &= \frac{1}{2a}\int_{-a}^{a} F_1(x)\mathrm{d}x, \quad A_n = \frac{1}{a}\int_{-a}^{a} F_1(x)\cos \alpha_n x \mathrm{d}x, \\ B_0 &= \frac{1}{2b}\int_{-b}^{b} F_2(y)\mathrm{d}y, \quad B_m = \frac{1}{b}\int_{-b}^{b} F_2(y)\cos \beta_m y \mathrm{d}y. \end{aligned} \tag{8}$$

Such a simple result is a consequence of the fact that the considered harmonic problem has simple homogeneous solutions. For other possible choices of the parameters α_n and β_m, the unknown coefficients in the solutions φ_I and φ_{II} have to be determined from an infinite set of algebraic equations.

The very simple boundary-value problem considered above gives the possibility to explain solution procedures for general boundary-value problems. Our example shows that this notion is more general than the homogeneous solutions of boundary-value problems. We will succeed in constructing an analytical solution of a boundary-value problem. It is important to note that every concrete problem has a number of general solutions. Practically we can use the form of solution that affords the simplest procedure of boundary-condition fulfillment. In the case of the harmonic problem this gives a way to obtain a homogeneous solution. But for other problems the situation looks more complicated.

At the same time the discussion shows that there are a lot of different forms of the general solution of the boundary-value problem. The use of these other solutions is not reasonable for the problem under consideration, but it is useful to understand more complicated situations as a basis to generate new approaches. We shall come across more complicated 2D boundary-value problems for elastic bodies.

To conclude we discuss the physical meaning of the functions ϕ_1 and ϕ_2. The sum of these functions forms the general solution (7) of the harmonic boundary-value problem on a rectangle. This fact is useful for understanding the ideas advanced by consideration of boundary-value problems in the theory of elasticity for bodies of finite size.

3. General solution of the 2D boundary-value problem for an elastic rectangle

Let us consider a 2D problem in elasticity for a rectangular domain (Figure 1). There is a difference in the physical meaning of two cases concerning a 2D problem in the theory of elasticity. One can distinguish states of plane strain and plane stress. Mathematically there is no difference between these two states. For the case where the stress is given on the surface, it is convenient to use a formulation of the 2D problem involving a stress function $\varphi(x, y)$ that satisfies the biharmonic equation

$$\frac{\partial^4\varphi}{\partial x^4} + 2\frac{\partial^4\varphi}{\partial x^2\partial y^2} + \frac{\partial^4\varphi}{\partial y^4} = 0. \tag{9}$$

The components of the stress tensor are determined by

$$\sigma_x = \frac{\partial^2\varphi}{\partial y^2}, \quad \sigma_y = \frac{\partial^2\varphi}{\partial x^2}, \quad \tau_{xy} = -\frac{\partial^2\varphi}{\partial x\partial y}. \tag{10}$$

To simplify the calculation, the symmetric stress state of the rectangle will be considered. It is assumed that the normal and tangent stress components are specified on the sides $x = \pm a$ and $y = \pm b$ as follows:

$$\begin{aligned} &\sigma_y(x, b) = F_1(x), \quad \tau_{xy}(x, b) = Q_1(x), \\ &\sigma_x(a, y) = F_2(y), \quad \tau_{xy}(a, y) = Q_2(y). \end{aligned} \tag{11}$$

The relations

$$\begin{aligned} &F_1(x) = F_1(-x), \quad Q_1(x) = -Q_1(-x), \\ &F_2(y) = F_2(-y), \quad Q_2(y) = -Q_2(-y) \end{aligned} \tag{12}$$

follow from the conditions of stress-state symmetry.

The particular solutions of the biharmonic equation can also be found by separation of variables. In the light of the harmonic problem discussed in the previous section, one can construct more simply the general solution of the boundary-value problem. Applying the logic used to construct the general solution for the harmonic problem, we obtain one part of the general solution from the following expression for the stresses:

$$\sigma_y^{(I)}(x, b) = \sum_{n=0}^{\infty} A_n \cos \gamma_n x, \quad \tau_{xy}^{(I)}(x, b) = \sum_{n=1}^{\infty} B_n \sin \gamma_n x, \quad \gamma_n = n\pi/l, \quad l \geq a. \tag{13}$$

Taking into account the expression for the stresses via the stress function, we easily find the first part of the general solution:

$$\varphi^{(I)}(x, y) = \sum_{n=0}^{\infty} \cos \gamma_n x [a_n \cosh \gamma_n y + b_n y \sinh \gamma_n y]. \tag{14}$$

This part of the solution contains two sequences of arbitrary constants, a_n, b_n, required to satisfy the boundary conditions on the edges $y = \pm b$. We can use the arbitrary value of the parameter γ_n to get a complete set trigonometric functions on these edges. The functions in terms of the coordinate y are obtained from an ordinary differential equation of the fourth order. The specific form of the functions corresponds to the symmetry of the stress state.

The second part of the solution is responsible for the fulfillment of the boundary conditions on the edges $x = \pm a$. The explicit presentation of the stress function is given by

$$\varphi^{(II)}(x, y) = \sum_{m=0}^{\infty} \cos \delta_m y [c_m \cosh \delta_m x + d_m x \sinh \delta_m x], \quad \delta_m = m\pi/p, \quad p \geq b. \tag{15}$$

Here c_m and d_m are arbitrary constants. The completeness of the representation of this part of the general solution exists for arbitrary $p \geq b$.

Thus we have obtained the solution

$$\varphi(x, y) = \varphi^{(I)}(x, y) + \varphi^{(II)}(x, y)$$

of the biharmonic equation. The process for the construction of the solution allows us to assert that the series obtained satisfies the biharmonic equation term by term and contain sufficient functional arbitrariness for the fulfillment of any boundary conditions. The construction procedure of such a solution for the biharmonic equation is not more difficult than for the harmonic equation.

Historically the idea used for the construction of the general solution is deeply rooted in the classical books by Lame [8, Lecture XII] and Mathieu [9, Chapter X]. But difficulties occurring in the procedure of boundary-condition fulfillment gave a stimulus to a number of different approaches to study the stress state in a rectangular domain. Many authors proposed some approximate solution providing a rudimentary answer for applied problems. One such problem was considered by Filon [10]. He proposed an approximate solution of the problem of compressing a finite elastic rectangle by concentrated forces. To calculate the stress in the domain he used only one part of the general solution presented here. This gave the possibility to satisfy exactly boundary conditions on the loaded surfaces only. The boundary conditions on the free surfaces were satisfied approximately (according to Saint-Venant's principle).

The difficulties in solving boundary-value problems for a rectangular elastic body, as met by Filon and other researchers, illustrates the principal difference between harmonic and biharmonic problems. In spite of the great simplicity of the general solution of the biharmonic problem given here, it is not possible to get simple homogeneous solutions in the elastic case. Let us consider the formula for the components of the stress tensor.

$$
\begin{aligned}
\sigma_x &= \sum_{n=0}^{\infty} \cos \gamma_n x [a_n \gamma_n^2 \cosh \gamma_n y + b_n \gamma_n (2 \cosh \gamma_n y + \gamma_n y \sinh \gamma_n y)] - \\
&- \sum_{m=0}^{\infty} \delta_m^2 \cos \delta_m y [c_m \cosh \delta_m x + d_m x \sinh \delta_m x], \\
\sigma_y &= -\sum_{n=0}^{\infty} \gamma_n^2 \cos \gamma_n x [a_n \cosh \gamma_n y + b_n y \sinh \gamma_n y] + \\
&+ \sum_{m=0}^{\infty} \cos \delta_m y [c_m \delta_m^2 \cosh \delta_m x + d_m \delta_m (2 \cosh \delta_m x + \delta_m x \sinh \delta_m x)], \\
\tau_{xy} &= -\sum_{n=0}^{\infty} \gamma_n \sin \gamma_n x [a_n \gamma_n \sinh \gamma_n y + b_n (\sinh \gamma_n y + \gamma_n y \cosh \gamma_n y)] - \\
&- \sum_{m=0}^{\infty} \delta_m \sin \delta_m y [c_m \delta_m \sinh \delta_m x + d_m (\sinh \delta_m x + \delta_m x \cosh \delta_m x)].
\end{aligned}
\tag{16}
$$

A simple analysis shows the difference between the biharmonic problem and the corresponding harmonic one. Considering, for example, the part of the expression for stress corresponding to the first part of the stress function $\varphi^{(I)}(x, y)$, one can see that it is not possible to make σ_x and τ_{xy} zero on the surface $x = a$ simultaneously for real values of γ_n. Here the general solution of the boundary-value problem does not transform directly to a sum of two homogeneous solutions.

We shall discuss the specific properties of homogeneous solutions of the biharmonic problem in the next section. It will be expedient to specify some examples showing the importance and usefulness of the concept of the general solution of boundary-value problems for the construction of the analytical solutions. The geometry of a 2D elastic domain, which will be considered as the first example, is shown in Figure 2. We consider the stress state of a rectangle with a circular cavity. The position of the cavity may be arbitrary. The symmetrical case is considered only for the sake of simplicity of formulation. If we want to get the solution of the biharmonic problem for such a domain, we have to construct first of all the general solution of the problem.

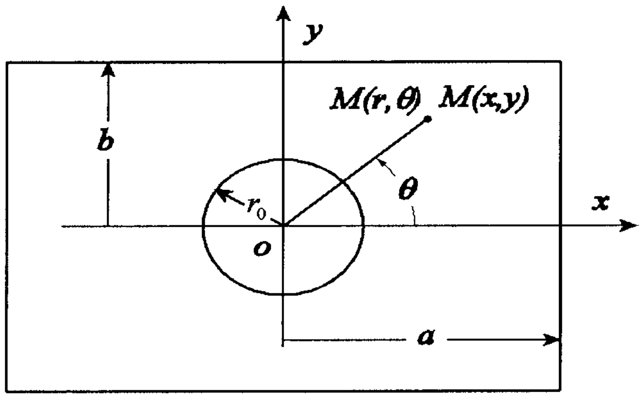

Figure 2. Example of a domain having general solution of boundary problem for the biharmonic equation.

As the first part of such a solution for this symmetrical case one can directly use the general solution for the solid rectangle given above. The biharmonic function $\varphi_1(x, u) = \varphi^{(I)}(x, y)+\varphi^{(II)}(x, y)$, were the terms are determined by Equations (14) and (15), contain sufficient functional arbitrariness to satisfy boundary conditions on straight parts of the boundary. To be fully confident regarding the possibility to satisfy boundary conditions on the circular cavity, we have to add to the biharmonic function $\varphi_1(x, y)$ the following biharmonic function

$$\varphi_2(x, y) = \varphi_2(r\ \cos\theta, r\ \sin\theta) = p_0 \log r+$$
$$+(p_1 r^{-1} + q_1 r \log r)\ \cos\theta + \sum_{k=2}^{\infty}(p_k r^{-k} + q_k r^{-k+2}) \cos k\theta. \tag{17}$$

The structure of this equation reflects the high level of symmetry of the stress state in the considered case. For more general cases one has to use the expression for the biharmonic function in polar coordinates given in [10]. By a suitable choice of the values of the arbitrary coefficients $p_0, p_k, q_k (k = 1, 2, \dots)$, one can satisfy arbitrary (symmetrical) conditions for radial and tangential stresses on a circular surface. Now other biharmonic boundary-value problems can be distinguished as being tractable within the scope of the discussed method. These may be different convex domains bonded by line segments and containing circular cavities. Of course, the practical calculation to get a quantitative estimate of the stress-field component may be not very simple and involve complicated infinite sets of algebraic equations. But is is important that the reduced series satisfy the biharmonic equation exactly and estimating the accuracy of the solution can easily be done by comparing calculated stress values with given boundary conditions. For some cases, as will be shown below, the method considered here yields a practically exact solution.

Finally, the following should be noted regarding the method of constructing the general solution of boundary-value problems for the biharmonic equation. The given general solution, being a superposition of the two biharmonic functions (14) and (15), illustrates that the idea of the general solution does not generate some unique analytical expression. In the specified Equations (14) and 15), the constants γ_n and δ_m can be quite arbitrary. In the harmonic case the unique form of the general solution arises when the general solution is transformed into a set of homogeneous ones. It is obvious, that the same procedure can be applied to a special case of boundary conditions for a biharmonic problem, when the stress function and its second normal derivative are given on the boundary of a rectangle.

It is possible to specify also some other types of boundary conditions, for which the homogeneous solution can be easily constructed for real values γ_n and δ_m in Equations (14) and (15). However, for general boundary-value problems there is no way to simplify the procedure of calculating the unknown coefficients by choosing γ_n and δ_m appropriately. When the stresses are specified on the boundary of a rectangle, it is convenient to use a quasi-homogeneous solution giving $\tau_{xy}^{I}(a, y) = 0$ and $\tau_{xy}^{II}(x, b) = 0$. This leads to the following parameter values:

$$\gamma_n = n\pi/a, \qquad \delta_m = m\pi/b. \tag{18}$$

Now an infinite set of algebraic equations has to be solved to determine the unknown coefficients of the series (14) and (15). The properties of the set and the solution algorithm will be considered below.

4. Homogeneous solutions for the biharmonic problem

The simplicity and gracefulness of the harmonic-problem solution in the scope of the method of homogeneous solutions stimulated much research towards the construction of homogeneous solutions of the biharmonic problem and a study of their properties.

The expressions for the stress components (16) in the constructed general solution of the biharmonic problem give a basis for finding homogeneous solutions. Let us consider the first part of the representation of the stress function (16) and the corresponding expressions for the stresses σ_y and τ_{xy} in (16)

$$\begin{aligned} \sigma_y &= -\sum_{n=0}^{\infty} \gamma_n^2 \cos \gamma_n x[a_n \cosh \gamma_n y + b_n y \sinh \gamma_n y], \\ \tau_{xy} &= -\sum_{n=0}^{\infty} \gamma_n \sin \gamma_n x[a_n \gamma_n \sinh \gamma_n y + b_n(\sinh \gamma_n y + \gamma_n y \cosh \gamma_n y)]. \end{aligned} \tag{19}$$

One can see that the surfaces $y = \pm b$ will be free ($\sigma_y = \tau_{xy} = 0$) if the separation constants γ_n satisfy the equations

$$\begin{aligned} &[a_n \cosh \gamma_n b + b_n b \sinh \gamma_n b] = 0, \\ &[a_n \gamma_n \sinh \gamma_n b + b_n(\sinh \gamma_n b + \gamma_n b \cosh \gamma_n b)] = 0. \end{aligned} \tag{20}$$

This being a homogeneous set of algebraic equations for the coefficients a_n and b_n, these equalities lead to the following equation for these separation constant γ_n:

$$\sinh 2\gamma_n b + 2\gamma_n b = 0. \tag{21}$$

This is the well-known equation for eigenvalues for the homogeneous solutions in the theory of elasticity [11].

Here it is necessary to pay attention to a basic difference in the approach to the construction of homogeneous solutions for harmonic and biharmonic problems. In the first case, using the first part of the representation for harmonic functions (5), we construct a complete system at the surface $y = \pm b$ and, by the choice of separation constants, satisfy zero boundary conditions at the surfaces $x = \pm a$. In the second case, we use completeness and orthogonality of the trigonometric functions cos $\gamma_n x$ and sin $\gamma_n x$ at the surfaces $y = \pm b$ to obtain a solution with zero values of normal and tangential stresses on this surface. Thus, the set of the particular solutions corresponding to the eigenvalues of Equation (21) allow to satisfy arbitrary boundary conditions for σ_x and τ_{xy} on the surfaces $x = \pm a$.

At first sight, the formal distinction results in the important basic properties of the solutions. First of all, it is necessary to pay attention to the fact that Equations (20) connect values of the constants a_n and b_n. As a result, the representation of the stresses σ_x and τ_{xy} by eigenfunctions of homogeneous solutions contains only one sequence of arbitrary coefficients. Qualitatively the solution of the problem is that Equation (21) has no real roots (except for trivial zero). Thus, the eigenvalues and eigenfunctions resulting from the homogeneous solutions are complex. The constants b_n should also be complex. In this connection the potential possibility is kept to satisfy two boundary conditions at the surfaces $x = \pm a$ by a suitable choice of real and imaginary parts of the constants b_n. Equation (21) for eigenvalues in the problem concerning the construction of homogeneous solutions for a boundary-value problem in the theory of elasticity (biharmonic problem) was obtained for the first time by Dougall [12]. Filon was the first researcher who realized the importance of the problem. His first memoir [2] was devoted to studying properties of eigenfunctions of the homogeneous solutions in 2D elasticity illustrating the essence of the problem, outlined a way towards its solution, and gave concrete examples for the representation of polynomial functions by series of such eigenfunctions. This brilliant work afforded a basis for many theoretical and applied investigations. The available studies can be divided into two main groups.

The first question arising from using the homogeneous-solutions method for the biharmonic problem concerns an important mathematical problem. When the separation constants γ_n are determined from Equation (21), the stresses σ_y and τ_{xy}, corresponding to the first biharmonic function (14), vanish on the surfaces $y = \pm b$. The idea behind the homogeneous-solution method is that the boundary conditions (11) on the surfaces $x = \pm a$ have to be satisfied by

$$a_n = b_n b \ \tan\ h\gamma_n b. \tag{22}$$

Substitution of this expression in the corresponding series in (16) gives the following functional equations for the boundary conditions (11):

$$\begin{aligned} &\sum_{n=0}^{\infty} b_n\gamma_n \cos\ \gamma_n a[\cosh\ \gamma_n y(2-\gamma_n b\ \tanh\ \gamma_n b) + \gamma_n y\ \sinh\ \gamma_n y] = F_2(y), \\ &-\sum_{n=0}^{\infty} b_n\ \sin\ \gamma_n a[\sinh\ \gamma_n y(1-\gamma_n b\ \tanh\ \gamma_n b) + \gamma_n y\ \cosh\ \gamma_n y] = Q_2(y). \end{aligned} \tag{23}$$

Mathematical substantiation of the ability to obtain the correct solution of this system of equations is connected to the proof of double multiplicity of eigenfunctions determined by the set of roots of the transcendental equation (21). That can be done within the scope of the spectral theory of non-self-adjoint operators [13, 14]. Extensive mathematical investigations have shown that, in principle, it is possible to solve these equations. For the concrete case of the biharmonic problem the corresponding mathematical theorems were proved by scientists of Vorovich's scientific school [15, 16] in Rostov (Russia).

It is difficult to understand why so much effort to solve the biharmonic problem within the scope of the method of homogeneous solutions is expended, when so simple and so effective a solution as that of Mathieu exists. Nevertheless, much work on the development of methods for the solution of Equation (21) has been performed.

Very interesting results were obtained by Papkovich, which gave a basis for considering the homogeneous-solutions method as a practical approach to getting quantitative characteristics of the stress field. The discussion of the biharmonic problem for a rectangular domain was given by Papkovich in [17]. He did much to popularize the method and for the development of procedures for the determination of the coefficients similar to the procedure of Fourier. This idea is very attractive and, in spite of the presence of mathematical problems in proving the solvability of the system of functional equations (23), the method of homogeneous solutions has often been used for the solution of concrete problems [19–21] and the development of approximate theories for plates [22]. Methods of finding roots of Equation (22) have been developed. The development of methods to transform the functional equations into algebraic ones is also important. The practical use of the method of homogeneous solutions is often based on collocation or mean-square-error minimization for the purpose of satisfying the boundary conditions (see [23]). Such expansions have properties that are essentially different from those of Fourier series. In the case of mixed-boundary-value problems, singularities arise at corner points. This requires the series of eigenfunctions of homogeneous solutions to be divergent on a finite interval (see [22]).

5. Important feature of the method of superposition

In the second section of this paper the general solution of the biharmonic boundary-value problem was constructed and analyzed. Taking into account the structure of the general solution $\varphi_1(x, y) = \varphi^{(I)}(x, y)+\varphi^{(II)}(x, y)$ one can give a physical interpretation of this representation. Every part of this sum is the general solution for a periodically deformed elastic strip. That gave to this method the name of method of superposition. Such physical reasoning was used by Lamé when he discussed the classical problem of an elastic parallelepiped in equilibrium. He immediately noted [8] that superposition of three solutions for mutually perpendicular elastic layers gives the solution of the problem of a parallelepiped in equilibrium. The understanding of the fact that the actual satisfaction of the boundary conditions gives rise to an infinite system of algebraic equations stopped subsequent development of this approach.

The simplest analysis of the expressions (16) lends basis to the conclusion about the possibility of satisfying arbitrary boundary conditions on the stresses on the sides of a rectangle. To arrive at this conclusion we only use well-known properties of the Fourier series. It is important to develop an effective procedure for the numerical implementation of the general formula.

First of all one can see that the separation constants γ_n and δ_m in the general solution are not determined uniquely. There are a number of ways to utilize this lack of uniqueness. The simplest way is to construct partially homogeneous solutions. One can see from formula (16) that the choice $\gamma_n = n\pi/a$ and $\delta_m = m\pi/b$ gives the possibility to satisfy boundary conditions for the tangential stresses on the surfaces $x = \pm a$, $y = \pm b$ independently. The choice $\gamma_n = (2n+1)\pi/(2a)$ and $\delta_m = (2m+1)\pi/(2b)$ gives the solution with such properties for the normal stresses. The detailed analysis will be given below for the first case.

The boundary conditions for the tangential stresses in (11) with respect to expressions (16) lead to the following functional equations

$$
\begin{aligned}
&-\sum_{n=0}^{\infty}\gamma_n \sin \gamma_n x[a_n\gamma_n \sinh \gamma_n b + b_n(\sinh \gamma_n b + \gamma_n b \cosh \gamma_n b)] = Q_1(x),\\
&-\sum_{m=0}^{\infty}\delta_m \sin \delta_m y[c_m\delta_m \sinh \delta_m a + d_m(\sinh \delta_m a + \delta_m a \cosh \delta_m a)] = Q_2(y).
\end{aligned}
\tag{24}
$$

Solution of these equations is a simple problem in mathematical analysis and after expanding the function in the right part in a Fourier series, we obtain the equations

$$
\begin{aligned}
a_n &= -b_n\left(\frac{1}{\gamma_n} + b \coth \gamma_n b\right) - \frac{1}{\gamma_n \sinh \gamma_n b} q_{1n},\\
c_m &= -d_m\left(\frac{1}{\delta_m} + a \coth \delta_m a\right) - \frac{1}{\delta_m \sinh \delta_m a} q_{2m},
\end{aligned}
\tag{25}
$$

where q_{1n} and q_{2m} are Fourier coefficients of the functions $Q_1(x)$ and $Q_2(y)$ determining the distribution of the tangential stresses on the edges of the rectangle.

The boundary conditions for the normal stresses give rise to more complicated functional equations, namely

$$
\begin{aligned}
&-\sum_{n=0}^{\infty}\gamma_n^2 \cos \gamma_n x[a_n \cos \gamma_n b + b_n\gamma_n b \sinh \gamma_n b]+\\
&\sum_{m=0}^{\infty}(-1)^m[c_m\delta_m^2 \cosh \delta_m x + d_m\delta_m(2\cosh \delta_m x + \delta_m x \sinh \delta_m x)] = F_1(x),\\
&\sum_{n=0}^{\infty}(-1)^n[a_n\gamma_n^2 \cosh \gamma_n y + b_n\gamma_n(2\cosh \gamma_n y + \gamma_n y \sinh \gamma_n y)]-\\
&-\sum_{m=0}^{\infty}\delta_m^2 \cos \delta_m y[c_m \cosh \delta_m a + d_m\delta_m a \sinh \delta_m a] = F_2(y).
\end{aligned}
\tag{26}
$$

Now in the functional equations terms from both parts of the general solution, formed by the sum of the biharmonic functions (14) and (15), are present. However, each equation contains the Fourier series with arbitrary coefficients that provides a basic element for the exact satisfaction of the boundary conditions for sufficiently arbitrary functions $F_1(x)$ and $F_2(y)$.

To transform the functional equations to algebraic ones, it is natural to use well-known properties of Fourier series. To that end, it is necessary to use the following equations

$$
\begin{aligned}
\cosh \delta_m x &= \frac{\sinh \delta_m a}{\delta_m a} + \sum_{n=1}^{\infty}\frac{2\delta_m(-1)^n \sinh \delta_m a}{a(\gamma_n^2+\delta_m^2)}\cos \gamma_n x,\\
x \sinh \delta_m x &= \frac{\cosh \delta_m a}{\delta_m} - \frac{\sinh \delta_m a}{a\delta_m^2} + \sum_{n=1}^{\infty}\frac{2\delta_m(-1)^n \cosh \delta_m a}{(\gamma_n^2+\delta_m^2)}\cos \gamma_n x+\\
&\quad + \sum_{n=1}^{\infty}\frac{2(-1)^n(\gamma_n^2-\delta_m^2)\sinh \delta_m a}{a(\gamma_n^2+\delta_m^2)^2}\cos \gamma_n x.
\end{aligned}
\tag{27}
$$

The same equations can be written for hyperbolic functions of y in the second equation in (26). These equations give the possibility, by obvious means, to transform the functional equations (26) into Fourier series with respect to $\cos \gamma_n x$ and $\cos \delta_m y$. In view of the awkwardness of the expressions, we shall give here only the appropriate series for the first equation.

$$\sum_{n=0}^{\infty} b_n\gamma_n\left[\cosh\ \gamma_n b+\frac{\gamma_n b}{\sinh\ \gamma_n b}\right]\cos\ \gamma_n x+\sum_{n=0}^{\infty}\gamma_n^2 q_{1n}^{*}\cosh\ \gamma_n b\ \cos\ \gamma_n x+$$
$$+\sum_{m=0}^{\infty}(-1)^m d_m\delta_m\sum_{n=1}^{\infty}\frac{4\delta_m\gamma_n^2(-1)^n\sinh\ \delta_m a}{a(\gamma_n^2+\delta_m^2)^2}\cos\ \gamma_n x-$$
$$-\sum_{m=0}^{\infty}(-1)^m q_{2m}^{*}\delta_m^2\left[\frac{\sinh\ \delta_m a}{\delta_m a}+\sum_{n=1}^{\infty}\frac{2\delta_m(-1)^m\sinh\ \delta_m a}{a(\gamma_n^2+\delta_m^2)}\cos\ \gamma_n x\right]=$$
$$=\sum_{n=0}^{\infty}F_{1n}\ \cos\ \gamma_n x, \tag{28}$$

where $q_{1n}^{*}=q_{1n}/\gamma_n\ \sinh\ \gamma_n b$ and $q_{2m}^{*}=q_{2m}/\delta_m\ \sinh\ \delta_m a$. Some comments have to be made before this equation can be transformed into an algebraic system. These comments concern the case $n=0$. Using the arbitrariness of the constants b_n, we can introduce the new constant $B_0=\gamma_n b_n(n\to 0)$. Taking into account that q_{10} and q_{20} are equal to zero for reasons of field symmetry, we can write the equation for the unbalanced part of the normal stresses σ_y as

$$2B_0-\sum_{m=1}^{\infty}(-1)^m q_{2m}^{*}\delta_m^2\frac{\sinh\ \delta_m a}{\delta_m a}=F_{10}. \tag{29}$$

For other terms of the Fourier series the following consequence of the algebraic equations follows from Equation (28)

$$b_n\gamma_n\left[\cosh\ \gamma_n b+\frac{\gamma_n b}{\sinh\ \gamma_n b}\right]+\sum_{m=1}^{\infty}(-1)^m d_m\delta_m\frac{4\delta_m\gamma_n^2(-1)^n\sinh\ \delta_m a}{a(\gamma_n^2+\delta_m^2)^2}+$$
$$+\gamma_n^2 q_{1n}^{*}\cosh\ \gamma_n b-\sum_{m=1}^{\infty}(-1)^m q_{2m}^{*}\delta_m^2\frac{2\delta_m(-1)^m\sinh\ \delta_m a}{a(\gamma_n^2+\delta_m^2)}=F_{1n}. \tag{30}$$

Having done similar transformations with the second equation in (26), we may derive a conjugate infinite system for the determination of the constants x_n and y_m.

$$x_n=\frac{1}{\Delta(\gamma_n b)}\sum_{m=1}^{\infty}y_m\frac{4\gamma_n^2}{a^2(\gamma_n^2+\delta_m^2)^2}+\beta_n,$$
$$y_m=\frac{1}{\Delta(\delta_m a)}\sum_{n=1}^{\infty}x_n\frac{4\delta_m^2}{b^2(\gamma_n^2+\delta_m^2)^2}+\alpha_m, \tag{31}$$
$$\Delta(\xi)=\frac{1}{\xi}\left(\coth\ \xi+\frac{\xi}{\sinh^2\ \xi}\right).$$

The unknown parameters in this system are connected to the coefficients b_n and d_m by the equations

$$x_n=b_n(-1)^n\gamma_n^2 b\ \sinh\ \gamma_n b,\quad y_m=-d_m(-1)^m\delta_m^2 a\ \sinh\ \delta_m a. \tag{32}$$

The constant terms β_n and α_m, though awkward, can be written according to the procedure demonstrated in connection with Equation (30).

The traditionally used forms of the homogeneous-solution method also produce infinite systems of algebraic equations. But it is not possible to get a corresponding system in so simple and explicit a form as Equations (31). One particularly important advantage of the

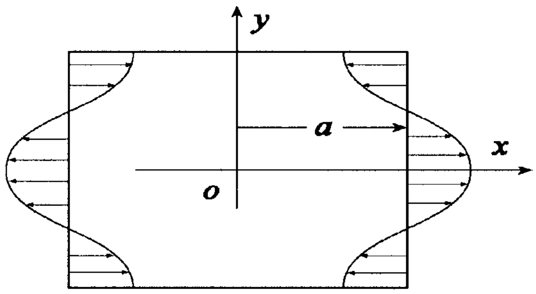

Figure 3. Scheme of loading of rectangle by sel-balanced forces.

method, called here the method of superposition, is that the system (31) can be investigated and solved within the framework of the theory of infinite systems put forward by Koialovich [7]. A detailed analysis of system (31), according to Koialovich's theory, was carried out in [24, Chapter 1]. The remarkable feature of Koialovich's theory is that it provides an algorithm for getting a practically exact estimate of all the infinitely many unknown parameters by solving a finite system.

To illustrate the possibilities of the method of superposition considered here, a specific boundary-value problem will be studied. To discuss some aspects of the Saint-Venant principle, the stress state of a square ($a = b$) caused by self-balanced normal forces will be studied. The functions in the boundary conditions (11) will have the following values

$$F_2(y) = \sigma_0 \cos \delta_M y, \qquad F_1(x) = Q_1(x) = Q_2(y) = 0. \tag{33}$$

Concrete quantitative data will be calculated for several leading values of M. The considered case for the first loading harmonic is shown in Figure 3. The infinite system for this specific geometry and loading obtains the form

$$\begin{aligned} x_n &= \frac{1}{\Delta(n\pi)} \sum_{m=1}^{\infty} y_m \frac{4n^2}{\pi^2(n^2+m^2)^2}, \qquad n = 1, 2, \dots \\ y_m &= \frac{1}{\Delta(m\pi)} \sum_{n=1}^{\infty} x_n \frac{4\delta_m^2}{\pi^2(n^2+m^2)^2} + \alpha_m, \qquad m = 1, 2, \dots \\ \alpha_m & \begin{cases} \sigma_0/\Delta(M\pi) & m = M, \\ 0, & m \neq M. \end{cases} \end{aligned} \tag{34}$$

The use of the procedure developed by Koialovich gives the possibility to get lower and upper bounds for the unknowns in an infinite system by solving a finite system. When the finite system, containing ten unknowns, has been solved, the obtained estimates are those presented in Table 1. The data correspond to the case $M = 1$. Here $\tilde{x}_n$ and $\tilde{y}_n$ give lower bounds for the corresponding parameters. Correspondingly, X_n and Y_n are the upper bounds. The estimate x_n^* and y_n^* were obtained by the method of simple reduction when the infinite system was replaced by a finite system with 20 unknowns. The values $\bar{x}_n$ and $\bar{y}_n$ are used in the procedures of the quantitative estimate of the stress state. All values are normalized by the parameter σ_0. In what follows the quantitative data for the cases $M = 1, 2, 3$ will be presented.

In problems of this type the most interesting data concern the character of the stress state near the loaded end of the rectangle. Such data allow to understand more deeply the important

Table 1. Estimated values of the unknown coefficients in the infinite set (34).

n	$\tilde{x}_n$	X_n	$\tilde{y}_n$	Y_n	x_n^*	y_n^*	$\bar{x}_n$	$\bar{y}_n$
1	3·5410	3·5412	1·1692	1·1694	3·5374	1·1657	3·5411	1·1693
2	0·9424	0·9431	1·7340	1·7347	0·9272	1·7188	0·9428	1·7343
3	1·1484	1·1498	1·7088	1·7103	1·1141	1·6746	1·1492	1·7094
4	1·2586	1·2612	1·6395	1·6422	1·1982	1·5791	1·2601	1·6405
5	1·3229	1·3270	1·5861	1·5502	1·2301	1·4933	1·3254	1·5874
6	1·3628	1·3685	1·5489	1·5546	1·2323	1·4184	1·3664	1·5505
7	1·3886	1·3961	1·5230	1·5305	1·2168	1·3512	1·3836	1·5249
8	1·4058	1·4152	1·5046	1·5140	1·1802	1·2889	1·4123	1·5068
9	1·4176	1·4290	1·4912	1·5026	1·1569	1·2305	1·4257	1·4938
10	1·4258	1·4392	1·4813	1·4946	1·1198	1·1753	1·4355	1·4840
>10	1·4312	1·4928	1·4312	1·4928	0	0	1·4598	1·4598

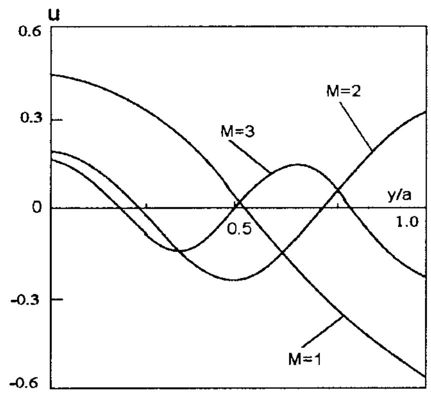

Figure 4. Distribution of displacements on the loaded surface.

features of the Saint-Venant principle. Let us first consider different cases of loading. The values of the normal to the loaded surface component of displacement vector are shown in Figure 4. Here $u = u_x\sigma_0/2aG$, where G is the shear modulus. The calculations were carried out for a Poisson ratio of $\nu = 1/3$.

Examining the data in Figure 4, we note that increasing the variability of external forces with constant amplitude results in decreasing displacement amplitudes. The elastic surface behaves as if it were more rigid. At the same time there is a significant difference between the values of the displacements in the center of the loaded surface $y = 0$; the displacement of the edge point almost twice exceeds that in the center point. Another interesting property of the stress state of the rectangle near the loaded sites gives information on the distribution of the stresses. The values of the normal stresses σ_y at $x = a$ are shown in Figure 5. The most important feature of these data is that the amplitudes of σ_y are essentially greater (by 25%) than the amplitude of the imposed stresses σ_x. This conclusion concerns all considered harmonic cases ($M = 1, 2, 3$). When M increases, the point with maximal value of the stresses drifts to the free surface $y = 0$. To characterize the stress state of a rectangle near the loaded surface it is important also to note information on the stress σ_x. The decaying character of

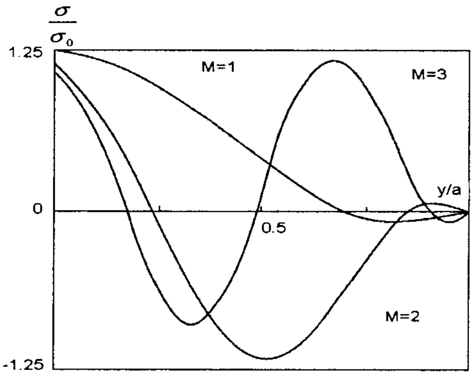

Figure 5. Distribution of the stress σ_y along loaded surface for the three cases of loading.

the stresses changes while moving away from the end: the decay rate along the unloaded surface $y = b$ is much slower than along the line $y = 0$. For example, the stress $\sigma_x(a, a)$ exceeds the stress $\sigma_x(a, 0)$ by 30%. As an illustration of a specific biharmonic problem, it is interesting to note that the stress $\sigma_x(0{\cdot}2a, a)$ exceeds the amplitude of the stress σ_x given by the boundary conditions. The analysis of the quantitative data concerning the stress state near the loaded surface of a rectangle permits to make an important comment concerning the Saint-Venant principle. This principle is often invoked to justify the application of approximate solutions for boundary-value problems with self-balanced loading on a part of the boundary. It is assumed that the values of the stresses caused by such self-balanced loadings do not exceed the maximum loading values anywhere in the interior. The data of our calculations on the practically exact solution for a rectangle show that this is not correct. Self-balanced loading can produce internal stresses exceeding the amplitudes of external loading. Understanding of this is important to realize the essence of the Saint-Venant principle.

6. Conclusion

Consideration of the solutions of boundary-value problems for harmonic and biharmonic equations forms a basis for explaining the idea of the general solution of the boundary-value problem. This idea provides a way of constructing effective analytical methods of solution for biharmonic boundary-value problems. To this end the properties of the well-studied eigenfunctions of harmonic problems are used to the full. A detailed consideration of harmonic and biharmonic problems for a rectangle has given a basis to clarify both common properties and the principal difference between two types of boundary-value problems.

The idea of the general solution is important to extend sufficiently the possibility to construct analytical solutions of boundary-value problems. One can use well-known particular solutions of biharmonic and other equations of mathematical physics in different coordinate systems to construct general solutions of complicated boundary-value problems. One specific example was given here, but ways of constructing many others are clear. The numerical implementation of such solutions is comparatively complicated, but the use of such solutions provides a simple way to control the accuracy of the solution.

The calculation for the case of a boundary-value problem for an elastic rectangle has demonstrated in principle the possibility that we may get practically exact solutions. It is possible to find an estimate for an infinite number of Fourier coefficients by solving a finite system of linear algebraic equations. As our calculations have shown, it is easy to attain an accuracy of 1–2% for these coefficients.

For specific cases of self-balanced loadings, qualitative data for the stresses have been given. It has been shown that self-balanced loadings can result in interior stresses exceeding the amplitudes of the imposed load. To realize this is important for the understanding of the salient aspect of the Saint-Venant principle and estimating errors arising from its application.

References

1. P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*. V.2 New York: McGraw-Hill (1953) 886 pp.
2. L. N. G. Filon, On the expansion of polynomials in series of functions, *Proc. London Math. Soc.* (ser.2) 4 (1907) 396–430.
3. G. V. Kolosov, *Application of Complex Diagrams and Theory of Functions of Complex Variable to the Theory of Elasticity*. Leningrad-Moscow: ONTI (1935) 224 pp. (in Russian).
4. N. I. Muskhelishvili, *Some Basic Problems of the Mathematical Theory of Elasticity* Groningen: Noordhoff (1963) 718 pp.
5. V. V. Meleshko, Selected topics in the history of the two-dimensional biharmonic problem. *Appl. Mech. Revs.* 56 (2003) 33–85.
6. I. G. Bubnov, *Structural Mechanics of a Ship*. Part II. St Petersburg: Naval Academy Press (1914) 310 pp. (in Russian).
7. B. M. Koialovich, Investigation on infinite systems of linear algebraic equations. *Trudy Steklov Physics and Mathematics Institute* 3 (1930) 41–167 (in Russian).
8. G. Lame, *Leçons sur la théorie mathématique de l'élasticité des corps solides*. Paris: Mallet-Bachelier (1852) 335 pp.
9. E. Mathieu, *Théorie de l'élasticité des corps solids*. Paris: Gauthier-Villars (1890) 403 pp.
10. S. P. Timoshenko and J. N. Goodier, *Theory of Elasticity*. 3rd edn (§43). New York: McGraw-Hill (1970) 567 pp.
11. A. I. Lourye, On the theory of thick plates. *Prikl. Mat. Mekh.* 6 (1942) 151–168 (in Russian).
12. J. Dougall, An analytical theory of the equilibrium of an isotropic elastic plate. *Trans. R. Soc. Edinburgh* 41 (1904) 129–228.
13. M. V. Keldysh, On completeness eigenfunctions of some classes of non self-adjoint linear operators. *Uspechi Mat. Nauk* 27 (1971) 15–47 (in Russian).
14. G. V. Radzievskiy, Problem of completeness of root vectors in spectral theory of operator-functions. *Uspechi Mat. Nauk* **37** (1982) 81–145 (in Russian).
15. I. I. Vorovich and V. E. Koval'chuk, On the basic properties of a system of homogenous solutions. *Prikl. Mat. Mekh.* 31 (1967) 861–870 (in Russian).
16. Yu. A. Ustinov and V. I. Yudovich, On the completeness of a system of elementary solutions of the biharmonic equation in a semi-strip. *Prikl. Mat. Mekh.* 37 (1973) 706–714 (in Russian).
17. P. F. Papkovich, *Structural Mechanics of a Ship*. 2, Leningrad: Sudpromgiz (1941) 960 pp. (in Russian).
18. G. G. Sebryakov, M. D. Kovalenko and N. N. Cybin, On some properties of the homogenous solution system in the theory of elasticity. *Dokl. Acad. Nauk Russia* 388 (2003) 193–196 (in Russian).
19. V. K. Prokopov, A survey of works on homogenous solutions in the theory of elasticity and their applications. *Trudy Leningrad. Politech. Inst.* No 279 (1967) 31–46 (in Russian).
20. B. M. Nuller, On solution of the theory of elasticity problem about equilibrium of truncated cone. *Izvestiya Acad. Science USSR. Mech. Solid Body* No 5 (1967) 17–29 (in Russian).
21. R. D. Gregory and J. Gladwell, The cantilever beam under tension, bending or flexure at infinity. *J. Elasticity* 12 (1982) 317–343.
22. O. K. Aksentyan, On stress concentration in thick plates. *Prikl. Mat. Mech.* 30 (1966) 963–970 (in Russian).
23. R. D. Gregory and J. Gladwell, The reflection of a symmetric Rayleigh-Lamb wave at fixed or free edge of a plate. *J. Elasticity* 20 (1984) 12–16.
24. V. T. Grinchenko, *Equilibrium and Steady Vibration of Elastic Bodies of Finite Dimensions* Kiev: Naukova Dumka (1978) 264 pp. (in Russian).